Transport Theory and Statistical Physics

ISSN: 0041-1450 (Print) 1532-2424 (Online) Journal homepage: http://www.tandfonline.com/loi/ltty20

SPECTRUM OF THE TRANSPORT OPERATOR IN A NONUNIFORM SLAB WITH GENERALIZED BOUNDARY CONDITIONS

Zhifeng Kuang , Mingzhu Yang & Guangtian Zhu

MARCEL DEKKER, INC. • 270 MADISON AVENUE • NEW YORK, NY 10016

TRANSPORT THEORY AND STATISTICAL PHYSICS
Vol. 31, No. 3, pp. 273–287, 2002

SPECTRUM OF THE TRANSPORT OPERATOR IN A NONUNIFORM SLAB WITH GENERALIZED BOUNDARY CONDITIONS

Zhifeng Kuang,[1,*] Mingzhu Yang,[1] and Guangtian Zhu[2]

[1]China Institute of Atomic Energy
P.O. Box 275(18), Beijing 102413
[2]Institute of Systems Science Academia Sinica,
Beijing 100080

ABSTRACT

Spectral properties of the transport operator in a nonuniform slab with generalized boundary conditions were first studied in Ref. [1]. The author showed that the operator is an infinitesimal generator of a C_0-semigroup and it has at least one real eigenvalue displaying the asymptotic behavior of the initial-value problem. Both continuous spectrum and possible accumulation points of isolated eigenvalues have not been considered. In this paper, we show that the continuous spectrum fill in a passage, and the possible accumulation points of isolated eigenvalues in the right half plane lie on the right boundary of the passage.

*E-mail: kuang@mipsa.ciae.ac.cn

DOI: 10.1081/TT-120015420
0041-1450 (Print); 1532-2424 (Online)
www.dekker.com

1. INTRODUCTION

Since the pioneering work of Lehner and Wing,[2,3] the spectrum of the neutron transport operator has been the subject of intensive study by mathematicians, physicists, and nuclear engineers. Instead of summarizing all the research, we concentrate ourselves on reviewing some relative work about the slab transport operator. Reference [2] treats mono-energetic neutron transport with isotropic scattering in a uniform slab surrounded either by vacuum or by a perfect absorber. And complete knowledge of the spectrum, a half plane of continuous spectrum plus finite real eigenvalues, is obtained. References [4,5] respectively extend the Lehner and Wing approach to uniform slab with anisotropic scattering and nonuniform slab with isotropic scattering. And knowledge of the spectrum, a half plane of continuous spectrum plus at least one real eigenvalue, is obtained. Reference [6] points out that the vacuum boundary conditions are not the only ones that need investigation and proposes the perfect reflection boundary conditions. The immitation of Lehner and Wing approach leads to a wrong assertion of Theorem 1 of Ref. [6]. The correct knowledge of the spectrum, a line of continuous spectrum plus finite real eigenvalues, is not obtained until in Ref. [7]. But the Fourier analysis method employed in Ref. [7] is useful only for uniform parallel shapes with perfect reflection boundary conditions. So it seems that both Lehner and Wing approach and Fourier analysis method are not employed to obtain complete knowledge of the spectrum of the operator in a nonuniform slab with generalized boundary conditions proposed in Ref. [1].

In this paper, first we introduce a transform which makes out the relation between the generalized boundary conditions and the perfect reflection conditions. Then we show that continuous spectrum of the streaming operator fills in a passage by using convex set theory and constructing a set of functions which is different from the one in Refs. [2,4,5]. Last we study the spectrum of the transport operator by perturbation theory for linear operators, and show that the left half plane is contained in the resolvent set and the possible accumulation points in the right half plane lie in the right boundary of the passage.

2. FORMULATION OF PROBLEMS

Let $H \equiv L^2(R)$ be the space of square-integrable functions in rectangle $R \equiv \{(x, \mu) : |x| \le a, |\mu| \le 1\}$ with the scalar product

$$(f, g) = \int_{-1}^{+1} d\mu \int_{-a}^{+a} dx f(x, \mu)\overline{g(x, \mu)},$$

and the norm $\|f\| = (f, f)^{1/2}$.

The transport operator in a nonuniform slab with generalized boundary conditions is defined in H by [cf. 1]:

$$A\bullet = B\bullet + \gamma(x)J\bullet, \quad B\bullet = -\mu\frac{\partial}{\partial x}\bullet - \sigma(x)\bullet,$$
$$J\bullet = \frac{1}{2}\int_{-1}^{+1}\bullet\, d\mu'.$$

The domain $D(J)$ of J is H, and the domains $D(A)$ and $D(B)$ of A and B are defined by $D(A) \equiv D(B) \equiv \{f \in H | f(x,\mu)$ is absolutely continuous in x for almost each fixed μ, $Bf \in H$, and $\delta(\mu)f(-a,\mu) = \eta(\mu)f(a,\mu)$, $|\mu| \le 1\}$.

$\sigma(x)$ is the total cross section, and $\gamma(x) = \sigma(x)C(x)$ where $C(x)$ is the mean number of secondaries per collision, and $\delta(\mu)$ and $\eta(\mu)$ are reflection functions.

Throughout this paper, we assume

(A1) $\sigma(x)$ and $\gamma(x)$ are nonnegative continuous functions in $[-a, a]$;

(A2) $\delta(\mu)$ and $\eta(\mu)$ are continuous functions in $[-1, 1]$ satisfying the reflection equation $\delta(-\mu) = \eta(\mu)$, $|\mu| \le 1$, and

$$0 < \eta_0 \le \eta(\mu) \le \delta(\mu) \le \eta_1 \le 1, \quad \mu \in (0, 1], \tag{2.1}$$

From the reflection equation and Eq. (2.1), one may easily see that

$$0 < \eta_0 \le \delta(\mu) \le \eta(\mu) \le \eta_1 \le 1, \quad \mu \in [-1, 0). \tag{2.2}$$

Let $f \in D(A)$. Considering the following transform

$$g(x,\mu) = \frac{1}{\delta(\mu)(a+x) + \eta(\mu)(a-x)} f(x,\mu), \tag{2.3}$$

we may find that $g \in \Theta \equiv \{g \in H | g(x,\mu)$ is absolutely continuous in x for almost each fixed μ, $\mu(\partial g/\partial x) \in H$, and $g(-a,\mu) = g(a,\mu), |\mu| \le 1\}$. And

$$\begin{aligned} Af &= -\mu\frac{\partial f}{\partial x} - \sigma(x)f + \frac{\gamma(x)}{2}\int_{-1}^{+1} f(x,\mu')\, d\mu' \\ &= [\delta(\mu)(a+x) + \eta(\mu)(a-x)] \\ &\quad \times \left\{-\mu\frac{\partial g}{\partial x} - U(x,\mu)g + \int_{-1}^{+1} k(x,\mu,\mu')g(x,\mu')\, d\mu'\right\} \end{aligned}$$

where

$$U(x,\mu) = \sigma(x) + \frac{\mu(\delta(\mu) - \eta(\mu))}{\delta(\mu)(a+x) + \eta(\mu)(a-x)} \tag{2.4}$$

$$k(x,\mu,\mu') = \frac{\gamma(x)}{2} \times \frac{\delta(\mu')(a+x) + \eta(\mu')(a-x)}{\delta(\mu)(a+x) + \eta(\mu)(a-x)} \tag{2.5}$$

Let us define the assistant operator T in H by:

$$T\bullet = L\bullet + K\bullet, \qquad L\bullet = -\mu\frac{\partial}{\partial x}\bullet - U(x,\mu)\bullet,$$
$$K\bullet = \int_{-1}^{+1} k(x,\mu,\mu')\bullet d\mu'.$$

The domains $D(T)$ and $D(L)$ of T and L are defined by the linear manifold Θ. Then

$$Af = [\delta(\mu)(a+x) + \eta(\mu)(a-x)]Tg \tag{2.6}$$

where f and g satisfy Eq. (2.3).

Remark 2.1. In the above, we refer ''continuous spectrum'' as the spectrum filling in a domain in C, the complex plane, it is a concept opposite to isolated spectrum. In the following, we follow the standard mathematical division and denote the resolvent set, the residual spectrum, continuous spectrum and point spectrum by $\rho(\bullet)$, $R_\sigma(\bullet)$, $C_\sigma(\bullet)$, and $P_\sigma(\bullet)$.

Theorem 2.1.

(1) $P_\sigma(A) = P_\sigma(T)$;
(2) $R_\sigma(A) = R_\sigma(T)$;
(3) $C_\sigma(A) = C_\sigma(T)$;
(4) $\sigma(A) = \sigma(T)$.

Proof.

(1) Comes directly from Eqs. (2.3) and (2.6).

(2) Suppose $\lambda \in R_\sigma(A)$. Then the range $R(\lambda - A)$ of $(\lambda - A)$ is not dense in H. This leads to the existence of such an $0 \neq f_0 \in H$ that $((\lambda - A)f, f_0) = 0$, for every f in $D(A)$.

Considering the function $g_0(x,\mu) = [\delta(\mu)(a+x) + \eta(\mu)(a-x)]f_0(x,\mu)$, we may find that $0 \neq g_0 \in H$, and for every $g \in D(T)$,

$$\begin{aligned}
&((\lambda - T)g, g_0)\\
&\quad = \lambda \int_{-a}^{a}\int_{-1}^{1} g(x,\mu)[\delta(\mu)(a+x) + \eta(\mu)(a-x)]\,\overline{f_0(x,\mu)}\,dx\,d\mu\\
&\quad + \int_{-a}^{a}\int_{-1}^{1} \mu\frac{\partial g}{\partial x}[\delta(\mu)(a+x) + \eta(\mu)(a-x)]\,\overline{f_0(x,\mu)}\,dx\,d\mu\\
&\quad + \int_{-a}^{a}\int_{-1}^{1}\left\{\sigma(x) + \frac{\mu[\delta(\mu) - \eta(\mu)]}{\delta(\mu)(a+x) + \eta(\mu)(a-x)}\right\}\\
&\quad \times g(x,\mu)[\delta(\mu)(a+x) + \eta(\mu)(a-x)]\,\overline{f_0(x,\mu)}\,dx\,d\mu
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{-a}^{a}\int_{-1}^{1}\gamma(x)\left[\int_{-1}^{1}\frac{\delta(\mu')(a+x)+\eta(\mu')(a-x)}{\delta(\mu)(a+x)+\eta(\mu)(a-x)}g(x,\mu')\,d\mu'\right]\\
&\times[\delta(\mu)(a+x)+\eta(\mu)(a-x)]\,\overline{f_0(x,\mu)}\,dx\,d\mu\\
&=\lambda\int_{-a}^{a}\int_{-1}^{1}G(x,\mu)\,\overline{f_0(x,\mu)}\,dx\,d\mu+\int_{-a}^{a}\int_{-1}^{1}\mu\frac{\partial G}{\partial x}\,\overline{f_0(x,\mu)}\,dx\,d\mu\\
&+\int_{-a}^{a}\int_{-1}^{1}\sigma(x)G(x,\mu)\,\overline{f_0(x,\mu)}\,dx\,d\mu\\
&-\frac{1}{2}\int_{-a}^{a}\int_{-1}^{1}\gamma(x)\int_{-1}^{1}G(x,\mu')d\mu'\,\overline{f_0(x,\mu)}\,dx\,d\mu\\
&=((\lambda-T)G,f_0)=0.
\end{aligned}
$$

This is because $[\delta(\mu)(a+x)+\eta(\mu)(a-x)]g(x,\mu)=G\in D(A)$. Therefore we show that $\lambda\in R_\sigma(T)$. The opposite procedure shows the inverse inclusion.

(3) Suppose $\lambda\in C_\sigma(A)$. Then there exists a set of functions $f_\tau\in D(A)$ such that $\|f_\tau\|\geq const>0$, and

$$\lim_{\tau\to 0}\|(\lambda-A)f_\tau\|=0.$$

Considering the function set

$$g_\tau(x,\mu)=\frac{1}{\delta(\mu)(a+x)+\eta(\mu)(a-x)}f_\tau(x,\mu),$$

we may find from Eqs. (2.3) and (2.6) that $g_\tau\in D(T)$, and

$$
\begin{aligned}
\|g_\tau\|^2&=\int_{-a}^{a}\int_{-1}^{1}\frac{1}{[\delta(\mu)(a+x)+\eta(\mu)(a-x)]^2}|f_\tau(x,\mu)|^2\,dx\,d\mu\\
&\geq\frac{1}{4a^2\eta_1^2}\|f_\tau\|^2\geq const>0
\end{aligned}
$$

$$
\begin{aligned}
\left\|(\lambda-T)g_\tau\right\|^2&=\int_{-a}^{a}\int_{-1}^{1}\frac{1}{[\delta(\mu)(a+x)+\eta(\mu)(a-x)]^2}|(\lambda-A)f_\tau(x,\mu)|^2\,dx\,d\mu\\
&\leq\frac{1}{4a^2\eta_0^2}\|(\lambda-A)f_\tau\|^2
\end{aligned}
$$

Hence $\lim_{\tau\to 0}\|(\lambda-T)g_\tau\|=0$. This shows $\lambda\in C_\sigma(T)$ after consideration of (1). And the same reason shows the inverse inclusion.

(4) is the direct conclusion of (1), (2) and (3).

Since all the proof of Theorem 2.1 is valid while $\gamma(x)\equiv 0$, we have the following corollary.

Corollary 2.1.

(1) $P_\sigma(B) = P_\sigma(L)$;
(2) $R_\sigma(B) = R_\sigma(L)$;
(3) $C_\sigma(B) = C_\sigma(L)$;
(4) $\sigma(B) = \sigma(L)$.

3. SPECTRUM OF L

It is easy to check under the reflection condition $\delta(-\mu) = \eta(\mu), |\mu| \le 1$ that the adjoint operator of A is as follows.

$$A^{*}\bullet = B^{*}\bullet + \gamma(x)J\bullet, \qquad B^{*}\bullet = \mu\frac{\partial}{\partial x}\bullet - \sigma(x)\bullet,$$

The domains $D(A^*)$ and $D(B^*)$ of A^* and B^* are defined by $D(A^*) \equiv D(B^*) \equiv \{\phi \in H | \phi(x, \mu)$ is absolutely continuous in x for almost each fixed μ, $B^*\phi \in H$, and $\eta(\mu) f(-a, \mu) = \delta(\mu) f(a, \mu)$, $|\mu| \le 1\}$.

Theorem 3.1.

(1) $R_\sigma(A) = R_\sigma(T) = \emptyset$;
(2) $R_\sigma(B) = R_\sigma(L) = \emptyset$.

Proof. Suppose $\lambda \in R_\sigma(A)$. Then $\overline{\lambda} \in P_\sigma(A^*)$ by [(11-7), P.244, 8]. Take $\phi \in D(A^*)$ such that $\overline{\lambda}\phi = A^*\phi$. That is

$$\overline{\lambda}\phi(x, \mu) = \mu\frac{\partial\phi}{\partial x} - \sigma(x)\phi(x, \mu) + \frac{\gamma(x)}{2}\int_{-1}^{1}\phi(x, \mu')\,d\mu'$$

Let $f(x, \mu) = \overline{\phi(x, -\mu)}$. It is easy to find that $f \in D(A)$, and $\lambda f = Af$. This contradiction shows that $R_\sigma(A) = \emptyset$.

(2) is the particular case of (1) when $\gamma(x) = 0$.

Since $\sigma(x)$, $\delta(\mu)$, and $\eta(\mu)$ are continuous functions, we can assume that $U_M = \max\{U(x, \mu) : (x, \mu) \in [-a, a] \times [-1, 1]\}$, and $U_m = \min\{U(x, \mu) : (x, \mu) \in [-a, a] \times [-1, 1]\}$.

Theorem 3.2. $\sigma(L) = \{\lambda : -U_M \le \mathrm{Re}\lambda \le -U_m\}$

Proof. Let Pass $= \{\lambda : -U_M \le \mathrm{Re}\lambda \le -U_m\}$. Then Pass is a closed convex set. For any $\lambda \notin$ Pass, the distance d from λ to Pass is positive. By applying the technique of integration by parts, it is easy to check that for any $g \in D(L)$, $\|g\| = 1$,

$$\begin{aligned}
2\mathrm{Re}(Lg, g) &= (Lg, g) + (g, Lg) \\
&= \int_{-a}^{a}\int_{-1}^{1}\left[-\mu\frac{\partial g}{\partial x} - U(x,\mu)g(x,\mu)\right]\overline{g(x,\mu)}\,dx\,d\mu \\
&\quad + \int_{-a}^{a}\int_{-1}^{1} g(x,\mu)\left[-\mu\frac{\partial \bar{g}}{\partial x} - U(x,\mu)\,\overline{g(x,\mu)}\right]dx\,d\mu \\
&= -2\int_{-a}^{a}\int_{-1}^{1} U(x,\mu)|g(x,\mu)|^2\,dx\,d\mu
\end{aligned}$$

This shows that $(Lg, g) \in \mathrm{Pass}$, while $g \in D(L)$, and $\|g\| = 1$. Hence for any $g \in D(L)$, $\|g\| = 1$,

$$d \le |\lambda - (Lg, g)| = |((\lambda - L)g, g)| \le \|(\lambda - L)g\|.$$

Hence $d\|g\| \le \|(\lambda - L)g\|$, for any $g \in D(L)$. This shows that $(\lambda - L)^{-1}$ exists and is continuous. The emptiness of $R_\sigma(L)$ follows $\lambda \in \rho(L)$. Therefore $\sigma(L) \subset \{\lambda : -U_M \le \mathrm{Re}\lambda \le -U_m\}$.

On the other hand, let $\Delta(\mu) = (1/2a)\int_{-a}^{a} U(s,\mu)\,ds$. Then $\Delta(\mu)$ is continuous in [−1, 1] and $[U_m, U_M] = \{\Delta(\mu) : \mu \in [-1, 1]\}$, since $U(x,\mu)$ is continuous in $[-a, a] \times [-1, 1]$, and $U_m \le U(x,\mu) \le U_M$.

For any $0 < \mu_0 < 1$, and $\varepsilon > 0$, there exists $\min(\frac{1}{2}, \varepsilon) > \tau > 0$ such that $|\Delta(\mu) - \Delta(\mu_0)| < \varepsilon$, for any μ in $[\mu_0, \mu_0 + \tau) \subset (0, 1]$

Choosing the following set of functions:

$$g_\tau(x,\mu) = \exp\left[-\frac{a-x}{2a\mu}\int_{-a}^{a} U(s,\mu)\,ds - \frac{1}{\mu}\int_{-a}^{x} U(s,\mu)\,ds + i\frac{n\pi}{a}x\right]b_\tau(\mu),$$
$$n = 0, \pm 1, \pm 2, \ldots$$

where

$$b_\tau(\mu) = \begin{cases} \dfrac{1}{\sqrt{\tau}}, & \mu \in (\mu_0 + \tau^2, \mu_0 + \tau) \\ 0, & \text{otherwise} \end{cases}$$

we may find that $g_\tau \in D(L)$, and

$$\begin{aligned}
\|g_\tau\|^2 &= \frac{1}{\tau}\int_{-a}^{a} dx \int_{\mu_0+\tau^2}^{\mu_0+\tau} \exp\left[-\frac{a-x}{a\mu}\int_{-a}^{a} U(s,\mu)\,ds - \frac{2}{\mu}\int_{-a}^{x} U(s,\mu)\,ds\right]d\mu \\
&\ge \frac{1}{\tau}\int_{-a}^{a} dx \int_{\mu_0+\tau^2}^{\mu_0+\tau} \exp\left(\frac{-4aU_M}{\mu_0}\right)d\mu \\
&= 2a(1-\tau)\exp\left(\frac{-4aU_M}{\mu_0}\right) > a\exp\left(\frac{-4aU_M}{\mu_0}\right)
\end{aligned}$$

$$\begin{aligned}
&\left\|\left(\left(-\Delta(\mu_0) - i\frac{n\pi}{a}\mu_0\right) - L\right)g_\tau\right)\right\|^2 \\
&\quad \leq 2\left\{\int_{-a}^{a} dx \int_{\mu_0+\tau^2}^{\mu_0+\tau} [\Delta(\mu) - \Delta(\mu_0)]^2 |g_\tau(x,\mu)|^2\, d\mu \right. \\
&\qquad \left. + \int_{-a}^{a} dx \int_{\mu_0+\tau^2}^{\mu_0+\tau} \left[\frac{n\tau}{a}(\mu - \mu_0)\right]^2 |g_\tau(x,\mu)|^2\, d\mu \right\} \\
&\quad < 4a\varepsilon^2(1-\tau) + \frac{4n^2\pi^2}{a}\tau^2(1-\tau) \\
&\quad < \left(4a + \frac{4n^2\pi^2}{a}\right)\varepsilon^2.
\end{aligned}$$

Therefore $-\Delta(\mu_0) - i(n\pi/a)\mu_0 \in \sigma(L)$. The arbitrariness of $\mu_0 \in (0,1)$ and n follows $\{-\Delta(\mu) - i(n\pi/a)\mu : \mu \in (0,1),\ n = 0,\ \pm 1,\ \pm 2, \ldots\} \subset \sigma(L)$. The similar procedure shows that $\{-\Delta(\mu) - i(n\pi/a)\mu : \mu \in (-1,0),\ n = 0,\ \pm 1,\ \pm 2, \ldots\} \subset \sigma(L)$. The closeness of $\sigma(L)$ leads to the inverse inclusion $\{\lambda : -U_M \leq \mathrm{Re}\lambda \leq -U_m\} \subset \sigma(L)$. Therefore we complete the proof.

4. SPECTRUM OF *T*

In this section, we further discuss the spectrum of T. First we show that the left half plane is contained in the resolvent set of T. Finally we show the possible accumulation points of isolated eigenvalues in the right half plane lie in the line $\mathrm{Re}\lambda = -U_m$ by following the outline of Ref. [9]. However, very strict conditions imposed on $U(x,\mu)$ and $k(x,\mu,\mu')$ in Ref. [9] are not satisfied here, say, $k(x,\mu,\mu')$ cannot be written in the form of $k(x,\mu,\mu') = |\mu'|^{-\tilde{\delta}}\tilde{k}(x,\mu,\mu')$, $\mu' \neq 0$ as Ref. [9]. Where $\tilde{\delta}$ is a real constant less than 1, $\partial\tilde{k}/\partial\mu$ and $\partial\tilde{k}/\partial\mu'$, are uniformly bounded.

Lemma 4.1.

(1) *For every* λ *with* $\mathrm{Re}\lambda < -U_M$, $\|(L - \lambda I)^{-1}\| \leq (-\mathrm{Re}\lambda - U_M)^{-1}$;
(2) *Let* $\gamma_M = \max\{\gamma(x) : |x| \leq a\}$. *Then* $\|K\| \leq (\eta_1/\eta_0)\gamma_M$.

Proof.

(1) For every λ with $\mathrm{Re}\lambda < -U_M$, $g \in D(L)$,

$$\begin{aligned}
&\mathrm{Re}((L-\lambda)g, g) \\
&\quad = (-\mathrm{Re}\lambda - U_M)(g,g) + \int_{-a}^{a}\int_{-1}^{1} (U_M - U(x,\mu))|g(x,\mu)|^2\, dx\, d\mu \\
&\qquad - \frac{1}{2}\int_{-a}^{a}\int_{-1}^{1}\left[\mu\frac{\partial g}{\partial x}\,\overline{g(x,\mu)} + \mu\frac{\partial \bar{g}}{\partial x}\, g(x,\mu)\right] dx\, d\mu \\
&\quad \geq (-\mathrm{Re}\lambda - U_M)\|g\|^2,
\end{aligned}$$

so $\|(L-\lambda)g\|\,\|g\| \geq |((L-\lambda)g,g)| \geq |\mathrm{Re}((L-\lambda)g,g)| \geq (-\mathrm{Re}\lambda - U_M)\|g\|^2$
This shows that $\|(L-\lambda I)^{-1}\| \leq (-\mathrm{Re}\lambda - U_M)^{-1}$;

(2) Since

$$k(x,\mu,\mu') = \frac{\gamma(x)}{2} \times \frac{\delta(\mu')(a+x)+\eta(\mu')(a-x)}{\delta(\mu)(a+x)+\eta(\mu)(a-x)} \leq \frac{\gamma_M}{2}\frac{\eta_1}{\eta_0},$$

and

$$\begin{aligned}
\|kg\|^2 &= \int_{-a}^{a}\int_{-1}^{1}\left|\int_{-1}^{1} k(x,\mu,\mu')g(x,\mu')\right|^2 dx\,d\mu \\
&\leq \int_{-a}^{a}\int_{-1}^{1}\left(\int_{-1}^{1}\left|k(x,\mu,\mu')\right|^2 d\mu'\right)\left(\int_{-1}^{1}\left|g(x,\mu')\right|^2 d\mu'\right) dx\,d\mu \\
&\leq \gamma_M^2\left(\frac{\eta_1}{\eta_0}\right)^2\|g\|^2,
\end{aligned}$$

So $\|K\| \leq (\eta_1/\eta_0)\gamma_M$.

Theorem 4.1. *For every* λ *with* $\mathrm{Re}\lambda < -U_M - (\eta_1/\eta_0)\gamma_{\mathrm{M}}$, $\lambda \in \rho(T)$.

Proof. For every λ with $\mathrm{Re}\lambda < -U_M - (\eta_1/\eta_0)\gamma_{\mathrm{M}}$, $\|(L-\lambda)^{-1}K\| \leq \|(L-\lambda)^{-1}\|\,\|K\| < (-\mathrm{Re}\lambda - U_M)^{-1} \times (\eta_1/\eta_0)\gamma_M < 1$. So $T-\lambda = (L-\lambda)\times(I+(L-\lambda)^{-1}K)$ has bounded inverse operator. The emptiness of $R_\sigma(T)$ concludes that $\lambda \in \rho(T)$.

Lemma 4.2.[9] *For every* λ *with* $\mathrm{Re}\lambda > -U_m$, $\lambda I - L$ *is invertible and* $\|(\lambda I - L)^{-1}\| \leq (\mathrm{Re}\lambda + U_m)^{-1}$, *and*

$$(\lambda I - L)^{-1}\phi = \begin{cases} \left\{1-\exp\left[-\frac{1}{\mu}\int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]\right\}^{-1} \\ \quad \times(J_1\phi + J_2\phi), \quad \text{if } \mu > 0, \\ \left\{1-\exp\left[\frac{1}{\mu}\int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]\right\}^{-1} \\ \quad \times(J_3\phi + J_4\phi), \quad \text{if } \mu < 0. \end{cases}$$

where

$$\begin{aligned}
J_1\phi(x,\mu) &= \frac{1}{\mu}\int_{-a}^{x}\exp\left[-\frac{1}{\mu}\int_{x'}^{x}(\lambda + U(s,\mu))\,ds\right]\phi(x',\mu)\,dx', \\
J_2\phi(x,\mu) &= \frac{1}{\mu}\int_{x}^{a}\exp\left\{-\frac{1}{\mu}\left[\int_{x'}^{x}(\lambda + U(s,\mu))\,ds \right.\right. \\
&\quad \left.\left. + \int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]\right\}\phi(x',\mu)\,dx',
\end{aligned}$$

$$J_3\phi(x,\mu) = -\frac{1}{\mu}\int_x^a \exp\left[\frac{1}{\mu}\int_x^{x'}(\lambda + U(s,\mu))\,ds\right]\phi(x',\mu)\,dx',$$

$$J_4\phi(x,\mu) = -\frac{1}{\mu}\int_{-a}^x \exp\left\{\frac{1}{\mu}\left[\int_x^{x'}(\lambda + U(s,\mu))\,ds + \int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]\right\}\phi(x',\mu)\,dx',$$

Theorem 4.2.

(1) *For every* λ *with* $\mathrm{Re}\lambda > -U_m$, $K(\lambda I - L)^{-1}K$ *is a compact operator on* H;

(2) *Suppose* $\zeta > \theta > -U_m$. *Then for every* $\varepsilon > 0$, *a positive* β_0 *independent of* $\alpha \in [\zeta, \theta]$ *exists such that* $\|K(\lambda I - L)^{-1}K\| < \varepsilon$ *uniformly in* $\{\lambda = \alpha + i\beta : \alpha \in [\zeta,\theta], |\beta| > \beta_0\}$.

Proof. For every λ with $\mathrm{Re}\lambda > -U_m$, and $\phi \in H$, a tedious calculation follows that

$$K(\lambda I - L)^{-1}K\phi = \int_{-a}^{a}\int_{-1}^{1} K_\lambda(x,\mu,x',\mu')\phi(x',\mu')\,dx'\,d\mu'$$

where

$$\begin{aligned}
K_\lambda(x,\mu,x',\mu') &= \int_0^1 \frac{1}{t}\left\{1 - \exp\left[-\frac{1}{t}\int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]\right\}^{-1} \\
&\quad \times \{k(x,\mu,\chi t)k(x',\chi t,\mu')\exp\left[-\frac{\chi}{t}\int_{x'}^{x}(\lambda + U(s,\chi t))\,ds\right] \\
&\quad + k(x,\mu,-\chi t)k(x',-\chi t,\mu')\exp\left[\frac{\chi}{t}\int_{x'}^{x}(\lambda + U(s,-\chi t))\,ds \right.\\
&\quad \left.\left. - \frac{1}{t}\int_{-a}^{a}(\lambda + U(s,-\chi t))\,ds\right]\right\}dt \\
&= \sum_{n=0}^{\infty}\int_0^1 \frac{1}{t}\exp\left[-\frac{n}{t}\int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]\left\{k(x,\mu,\chi t)k(x',\chi t,\mu')\right. \\
&\quad \times \exp\left[-\frac{\chi}{t}\int_{x'}^{x}(\lambda + U(s,\chi t))\,ds\right] + k(x,\mu,\,-\chi t)k(x',-\chi t,\mu') \\
&\quad \left.\times \exp\left[\frac{\chi}{t}\int_{x'}^{x}(\lambda + U(s,-\chi t))\,ds - \frac{1}{t}\int_{-a}^{a}(\lambda + U(s,-\chi t))\,ds\right]\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(P_n + Q_n),\\
P_n &\equiv P_n(x, x', \mu, \mu', \alpha, \beta)\\
&= \int_0^1 \frac{1}{t}\exp\left[-\frac{n}{t}\int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]k(x,\mu,\chi t)k(x',\chi t,\mu')\\
&\quad\times \exp\left[-\frac{\chi}{t}\int_{x'}^{x}(\lambda + U(s,\chi t))\,ds\right],\\
Q_n &\equiv Q_n(x, x', \mu, \mu', \alpha, \beta)\\
&= \int_0^1 \frac{1}{t}\exp\left[-\frac{n}{t}\int_{-a}^{a}(\lambda + U(s,\mu))\,ds\right]k(x,\mu,\,-\chi t)k(x',-\chi t,\mu')\\
&\quad\times \exp\left[\frac{\chi}{t}\int_{x'}^{x}(\lambda + U(s,-\chi t))\,ds - \frac{1}{t}\int_{-a}^{a}(\lambda + U(s,-\chi t))\,ds\right]dt,
\end{aligned}
$$

and

$$
\chi = \begin{cases} 1, & \text{if } x \ge x',\\ -1, & \text{if } x < x'. \end{cases}
$$

(1) Let $k_M = (\gamma_M/2)(\eta_1/\eta_0)$. Then $|k(x,\mu,\mu')| < k_M$, and an integration by parts concludes

$$
\begin{aligned}
|P_n + Q_n| &\le 2k_M^2 \int_0^1 \frac{1}{t}\exp\left[-\frac{2na}{t}(\alpha + U_m)\right]dt\\
&= 2k_M^2 \int_1^{+\infty}\frac{1}{t}\exp[-2na(\alpha + U_m)t]\,dt\\
&\le -\frac{2k_M^2}{2an(\alpha + U_m)}\Bigg\{-\exp[-2na(\alpha + U_m)]\\
&\quad + \int_1^{+\infty}\frac{1}{t^2}\exp[-2an(\alpha + U_m)t]\,dt\Bigg\}\\
&< \frac{2k_M^2}{2a(\alpha + U_m)}\frac{\exp[-2na(\alpha + U_m)]}{n}\\
&\le \frac{2k_M^2}{2a(\zeta + U_m)}\frac{\exp[-2na(\zeta + U_m)]}{n}
\end{aligned}
$$

$$
\sum_{n=0}^{+\infty}|P_n + Q_n| \le \frac{2k_M^2}{2a(\zeta + U_m)}\sum_{n=0}^{+\infty}\frac{\exp[-2na(\zeta + U_m)]}{n} \equiv C_1 < +\infty. \quad (4.1)
$$

Therefore $\int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1}|K_\lambda(x,\mu,x',\mu')|^2\,dx\,d\mu\,dx'\,d\mu' < +\infty$. This shows that $K(\lambda I - L)^{-1}K$ is a compact operator in H.

In addition, for any $\varepsilon > 0$, there exists an integer N_0 such that

$$\sum_{n=N_0+1}^{+\infty} |P_n + Q_n| < \frac{\varepsilon^2}{48C_1\alpha^2} \tag{4.2}$$

(2) Suppose $\zeta > \theta > -U_m$, $\alpha \in |\zeta, \theta|$.

$$\begin{aligned}
&\|K(\lambda I - L)^{-1}K\|^2 \\
&\quad \le \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |K_\lambda(x,\mu,x',\mu')|^2\, dx\, d\mu\, dx' d\mu' \\
&\quad \le C_1 \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |K_\lambda(x,\mu,x',\mu')|\, dx\, d\mu\, dx' d\mu' \\
&\quad \le C_1 \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} \left(\sum_{n=0}^{N_0} |P_n| + \sum_{n=0}^{N_0} |Q_n| \right. \\
&\qquad \left. + \sum_{n=N_0+1}^{+\infty} |P_n + Q_n| \right) dx\, d\mu\, dx' d\mu' \\
&\quad = C_1 \sum_{n=0}^{N_0} \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |P_n|\, dx\, d\mu\, dx' d\mu' \\
&\qquad + C_1 \sum_{n=0}^{N_0} \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |Q_n|\, dx\, d\mu\, dx' d\mu' \\
&\qquad + C_1 \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} \left(\sum_{n=N_0+1}^{+\infty} |P_n + Q_n| \right) dx\, d\mu\, dx'\, d\mu',
\end{aligned} \tag{4.3}$$

By Eq. (4.2), the third term of the right side of Eq. (4.3) is less than $\varepsilon^2/3$. In the following, let's estimate the first term of the right side of Eq. (4.3).

Suppose α', $\alpha'' \in [\zeta, \theta]$, and $\alpha' < \alpha''$.

$$\begin{aligned}
&|P_n(x, x', \mu, \mu', \alpha', \beta) - P_n(x, x', \mu, \mu', \alpha'', \beta)| \\
&\quad = \left| \int_0^1 \frac{1}{t} k(x,\mu,\chi t) k(x',\chi t,\mu') \exp\left\{ -\frac{1}{t}\left[n\int_{-a}^{a} (U(s,\chi t) - U_m)\, ds \right.\right.\right. \\
&\qquad \left.\left. + \chi \int_{x'}^{x} (U(s,\chi t) - U_m)\, ds \right] \right\} \left\{ \exp\left[-\frac{2an + |x - x'|}{t}(\alpha' + U_m) \right] \right. \\
&\qquad \left.\left. - \exp\left[-\frac{2an + |x - x'|}{t}(\alpha'' + U_m) \right] \right\} \exp\left[\frac{-2an + |x - x'|}{t}\beta i \right] dt \right|
\end{aligned}$$

$$\begin{aligned}
&\leq k_M^2 \int_0^1 \frac{1}{t} \exp\left[-\frac{2an + |x - x'|}{t}(\alpha' + U_m)\right] \\
&\quad \times \left\{1 - \exp\left[-\frac{2an + |x - x'|}{t}(\alpha'' - \alpha')\right]\right\} dt \\
&= k_M^2 \int_{2an+|x-x'|}^{+\infty} \frac{1}{t} \exp[-(\alpha' + U_m)t]\{1 - \exp[-(\alpha'' - \alpha')t]\}\, dt \\
&\leq k_M^2 \int_{2an}^{+\infty} \frac{1}{t} \exp[-(\alpha' + U_m)t]\{1 - \exp[-(\alpha'' - \alpha')t]\}\, dt \\
&\leq k_M^2 \int_{T_0}^{+\infty} \frac{1}{t} \exp[-(\alpha' + U_m)t]\, dt + k_M^2 \int_{2an}^{T_0} \frac{1}{t} \exp[-(\alpha' + U_m)t] \\
&\quad \times \{1 - \exp[-(\alpha'' - \alpha')t]\}\, dt
\end{aligned}$$

From

$$\lim_{T_0 \to +\infty} \int_{T_0}^{+\infty} \frac{1}{t} \exp[-(\alpha' + U_m)t]\, dt = 0,$$

one can easily see that there exists $0 < \delta_0$, independent of β, such that

$$\begin{aligned}
&\left|P_n(x, x', \mu, \mu', \alpha', \beta) - P_n(x, x', \mu, \mu', \alpha'', \beta)\right| \\
&\quad < \frac{\varepsilon^2}{96a^2 C_1(N_0 + 1)}, \quad \text{as } |\alpha' - \alpha''| < \delta_0
\end{aligned} \tag{4.4}$$

Moreover, let $\zeta = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = \theta$, $|\alpha_i - \alpha_{i+1}| < \delta_0$, $i = 1, 2, \ldots, n-1$. Then

$$\begin{aligned}
&P_n(x, x', \mu, \mu', \alpha_i, \beta) \\
&\quad = \int_0^1 \frac{1}{t} k(x, \mu, \chi t) k(x', \chi t, \mu') \exp\left\{-\frac{1}{t}\left[n \int_{-a}^{a} (\alpha_i + U(s, \chi t))\, ds \right.\right. \\
&\quad\quad \left.\left. + \chi \int_{x'}^{x} (\alpha_i + U(s, \chi t))\, ds\right]\right\} \exp\left[-\frac{2an + |x - x'|}{t}\beta i\right] dt \\
&\quad = \int_{2an+|x-x'|}^{+\infty} \frac{1}{t} k\left(x, \mu, \frac{2an + |x - x'|}{t}\chi\right) k\left(x', \frac{2an + |x - x'|}{t}\chi, \mu'\right) \\
&\quad\quad \times \exp\left\{-\frac{t}{2an + |x - x'|}\left[n \int_{-a}^{a} \left(\alpha_i + U\left(s, \frac{2an + |x - x'|}{t}\chi\right)\right) ds \right.\right. \\
&\quad\quad \left.\left. + \ \chi \int_{x'}^{x} \left(\alpha_i + U\left(s, \frac{2an + |x - x'|}{t}\chi\right)\right) ds\right]\right\} \exp(-t\beta i)\, dt
\end{aligned}$$

This shows that for fixed x, x', μ, μ', α_i, P_n is a Fourier coefficient of an integrable function. So

$$\lim_{|\beta|\to+\infty} |P_n(x, x', \mu, \mu', \alpha_i, \beta)| = 0.$$

By Lebesgue convergence theorem,

$$\lim_{|\beta|\to+\infty} \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |P_n(x, x', \mu, \mu', \alpha_i, \beta)|\, dx\, d\mu\, dx' d\mu' = 0.$$

This shows that there exists a $\beta_0' > 0$, such that for any α_i, $1 \le i \le n$,

$$\int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |P_n(x, x', \mu, \mu', \alpha_i, \beta)|\, dx\, d\mu\, dx' d\mu' < \frac{\varepsilon^2}{6C_1(N_0+1)}, \text{ as } |\beta| > \beta_0', \tag{4.5}$$

Suppose $\alpha_{i_0} \le \alpha < \alpha_{i_0+1}$, $1 \le i_0 < n$. Then by Eqs. (4.4) and (4.5),

$$\begin{aligned}
&\int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |P_n(x, x', \mu, \mu', \alpha, \beta)|\, dx\, d\mu\, dx' d\mu' \\
&\quad \le \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |P_n(x, x', \mu, \mu', \alpha, \beta) - P_n(x, x', \mu, \mu', \alpha_{i_0}, \beta)|\, dx\, d\mu\, dx'\, d\mu' \\
&\qquad + \int_{-a}^{a}\int_{-1}^{1}\int_{-a}^{a}\int_{-1}^{1} |P_n(x, x', \mu, \mu', \alpha_{i_0}, \beta)|\, dx\, d\mu\, dx'\, d\mu' \\
&\quad < \frac{\varepsilon^2}{3C_1(N_0+1)}
\end{aligned}$$

So the first term of the right side of Eq. (4.3) is less than $\varepsilon^2/3$ as $|\beta| > \beta_0'$.

The same reason shows that there exists $\beta_0'' > 0$, independent of $\alpha \in |\zeta, \theta|$, such that the second term of the right side of Eq. (4.3) is less than $\varepsilon^2/3$ as $|\beta| > \beta_0''$.

Therefore for every $\varepsilon > 0$, a positive $\beta_0 = \max\{\beta_0', \beta_0''\}$ independent of $\alpha \in [\zeta, \theta]$ exists such that $\|K(\lambda I - L)^{-1}K\| < \varepsilon$ uniformly in $\{\lambda = \alpha + i\beta : \alpha \in [\zeta, \theta], |\beta| > \beta_0\}$.

Up to now, one can easily conclude our main theorem by [Theorem 5.35, p. 244, 10] and our Theorems 3.2 and 4.2.

Theorem 4.3. *The transport operator A decomposes the spectral plane λ as follows.*

(1) $\{\lambda : \mathrm{Re}\lambda < -U_M - (\eta_1/\eta_0)\gamma_M\} \subset \rho(A)$.

(2) $\{\lambda : -U_M \leq \mathrm{Re}\lambda \leq -U_m\} \subset \sigma(A)$.
(3) $\{\lambda : \mathrm{Re}\lambda > -U_M\} \cap \sigma(A)$ *contains at most countable isolated eigenvalues of A with finite algebraic multiplicity*, *and the possible accumulation points of them only appear on the line* $\mathrm{Re}\lambda = -U_m$.

ACKNOWLEDGMENTS

This work was supported by National Natural Science Foundation of China and the Science Foundation of Nuclear Industry of China.

REFERENCES

1. Belleni-Morante, A. J. Math. Phys. **1970**, *11*, 1553–1557.
2. Lehner, J.; Wing, G.M. Comm. Pure Appl. Math. **1955**, *8*, 217–234.
3. Lehner, J.; Wing, G.M. Duke Math. J. **1956**, *23*, 125–142.
4. Mika, J. J. Math. Phys. **1966**, *7*, 833–838.
5. Mika, J. J. Math. Phys. **1966**, *7*, 839–843.
6. Belleni-Morante, A. J. Math. Anal. and Appl. **1970**, *30*, 353–374.
7. Angelescu, N.; Marinescu, N.; Protopopescu, V. TTSP, **1976**, *5*(2&3), 115–125.
8. Taylor, A.E.; Lay, D.C. *Introduction to Functional Analysis*, Second Edition; John Wiley & Sons, Inc., 1980.
9. Degong Song. TTSP **1996**, *25*(7), 753–784.
10. Kato, L. *Perturbation Theory for Linear Operators*, Grundlehren der Mathematischen Wissenschaften, 132, Springer-Verlag, 1984.

Received April 21, 1997
Accepted May 31, 2001