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ON THE INDEX OF THE EIGENVALUE OF A TRANSPORT OPERATOR

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ABSTRACT

The index of the eigenvalue of the monoenergetic neutron transport operator is studied under the assumptions of homogeneity and boundedness of the medium and of isotropy of scattering. It is shown that all isolated real eigenvalues are with index one.

In addition, we show that there is no nonreal eigenvalue in $\{\lambda \in \mathbf{C} \mid \mathrm{Re}\lambda > -\Sigma\}$

I . INTRODUCTION

We consider a homogeneous bounded medium V with a convex surface Γ surrounded by vacuum. The production of neutrons by scattering and fission is assumed to be isotropic. Let U be the surface of unit sphere in R^3. $\mathbf{r}$ and Ω denote the position vector and the direction of neutron motions, respectively, $\mathbf{r} \in V$, $\Omega \in U$. Let $H \equiv L^2(G)$, $G = V \times U$, be the usual Hilbert space with the inner-product:

$$(f,g) = \int_V d\mathbf{r} \int_U f(\mathbf{r},\Omega)\, \overline{g(\mathbf{r},\Omega)}\, d\Omega, \quad f,g \in H \tag{1}$$

Now our eigenvalue problem can be formulated by $\lambda f = Af$. Where the operator A is defined by:

$$Af(\mathbf{r},\Omega) = -\Omega \cdot \mathrm{grad}_{\mathbf{r}} f(\mathbf{r},\Omega) - \Sigma f(\mathbf{r},\Omega) + \frac{C}{4\pi}\int_U f(\mathbf{r},\Omega')d\Omega' \qquad (2)$$

with its domain $D(A) \equiv \{f \in H \mid Af \in H;\ f(\mathbf{r},\Omega) = 0$ for $\mathbf{r} \in \Gamma$ and Ω entering direction to the body $V\}$.

Here Σ and C are positive constants.

According to the results in [1] [7], the spectrum $\sigma(A)$ of A consists of infinite isolated eigenvalues with finite algebraic multiplicities accumulating at $-\infty$.

For any eigenvalue λ of A, let $N((\lambda I - A)^p) = \{f \in H \mid (\lambda I - A)^p f \equiv 0\}$. Then we call the smallest integer p such that

$$N((\lambda I - A)^p) = N((\lambda I - A)^{p+1}) \qquad (3)$$

by the index $\alpha(\lambda)$ of λ.

In order to get the asymptotic expansion of the solution of the corresponding transport equation, one should study the index of the eigenvalue of A [8]. In the case of spherically symmetric scattering in a homogeneous sphere, R. Van Norton[2] showed that A has infinite real eigenvalues and all the real eigenvalues are with index one. Although Ukai[1] extended the existence of infinite real eigenvalues to the arbitrary homogeneous convex body, he failed in showing they are with index one. In section 2 of this paper, we show that all the real eigenvalues are with index one. In section 3, we show that there is no nonreal eigenvalue of A in $\{\lambda \in \mathbf{C} \mid \mathrm{Re}\lambda > -\Sigma\}$.

Ⅱ. INDEX OF REAL EIGENVALUES

In this section, we establish a general theorem (Theorem2. 1) which assures that one can consider the index of 1 as an eigenvalue of a compact operator instead of the index of the eigenvalue of A. Then by use of the theorem, we show that all the real eigenvalues of A are with index one.

Define operators L, K, J in H as follows:

$$Lf = -\ \Omega \cdot \mathrm{grad}_{\mathbf{r}}\, f(\mathbf{r},\Omega) - \Sigma f(\mathbf{r},\Omega) \tag{4}$$

$$Kf = \frac{C}{4\pi}\int_U f(\mathbf{r},\Omega')d\Omega' \tag{5}$$

$$Jf = \int_U f(\mathbf{r},\Omega')d\Omega' \tag{6}$$

with their domains $D(L)=D(A)$, $D(K)=D(J)=H$.

It is easy to know the adjoint operators L^* and K^* of L and K are defined by:

$$L^*f = \Omega \cdot \mathrm{grad}_{\mathbf{r}}\, f(\mathbf{r},\Omega) - \Sigma f(\mathbf{r},\Omega) \tag{7}$$

with its domain $D(L^*)\equiv\{f\in H \mid L^*f\in H;\ f(\mathbf{r},-\Omega)=0$ for $\mathbf{r}\in\Gamma$ and Ω entering direction to the body $V\}$.

$$K^*f = \frac{C}{4\pi}\int_U f(\mathbf{r},\Omega')d\Omega' \tag{8}$$

with its domain $D(K^*)=H$.

It is easy to know the following four lemmas.

Lemma 2.1[7]. L is a densely defined closed linear operator, and for any complex λ, $(\lambda I-L)^{-1}$ and $(\lambda I-L^*)^{-1}$ exist and are bounded in H.

Lemma 2.2[3][4]. Let $K_\lambda=(\lambda I-L)^{-1}K$, then K_λ is a compact operator from H to H.

Lemma 2.3. [Theorem10.1, p.330, 5] The index of an eigenvalue λ_0 of A is p if and only if it is a pole of its resolvent operator R_λ of order p.

Lemma 2.4. λ_0 is a pole of R_λ of order p if and only if λ_0 is a pole of $(I-(\lambda I-L)^{-1}K)^{-1}$ of order p.

Theorem 2.1. If for any complex λ, the unity is an eigenvalue of K_λ with index 1, then λ is an eigenvalue of A with index 1 also.

Proof. If for any complex λ_0, the unity is an eigenvalue of K_{λ_0} with index 1, then by the compactness of K_{λ_0},

$$R((I-(\lambda_0 I-L)^{-1}K)^2)=R(I-(\lambda_0 I-L)^{-1}K) \tag{9}$$

$$N(I-(\lambda_0 I-L)^{-1}K)\oplus R(I-(\lambda_0 I-L)^{-1}K)=H. \tag{10}$$

By Lemma 2.3 and Lemma 2.4, in order to show the theorem, it is enough to show that λ_0 is a pole of $(I-(\lambda I-L)^{-1}K)^{-1}$ of order 1.

Let

$$B_m=\frac{1}{2\pi i}\int_S (I-(\lambda I-L)^{-1}K)^{-1}(\lambda-\lambda_0)^{m-1}d\lambda, \quad m=1,2,\cdots \tag{11}$$

Where S is a counterclockwise circle $|\lambda-\lambda_0|=\delta$, in $|\lambda-\lambda_0|\leqslant\delta$, $(I-(\lambda I-L)^{-1}K)^{-1}$ is analytic except for λ_0.

For any $x\in R(I-(\lambda_0 I-L)^{-1}K)$, let $F(\lambda)=(I-(\lambda I-L)^{-1}K)^{-1}x$.

Since one easily knows that $I-(\lambda_0 I-L)^{-1}K$ is bijective on the closed subspace $R(I-(\lambda_0 I-L)^{-1}K)$ from (9) and (10), λ_0 is a removable singular point of $F(\lambda)$. Therefore

$$B_1 x=0, \text{ for any } x\in R\,(I-(\lambda_0 I-L)^{-1}K). \tag{12}$$

For any $f\in D(A)$ and $\lambda\in S$, $(I-(\lambda I-L)^{-1}K)^{-1}f\in D(A)$. Since A is closed,

$$\begin{aligned}(\lambda_0-L-K)B_1 f&=\frac{1}{2\pi i}\int_S(\lambda_0-L-K)(I-(\lambda I-L)^{-1}K)^{-1}fd\lambda\\&=\frac{1}{2\pi i}\int_S(I-(\lambda I-L)^{-1}K)^{-1}(-\lambda+\lambda_0)f+(\lambda I-L)fd\lambda\\&=\frac{1}{2\pi i}\int_S(I-(\lambda I-L)^{-1}K)^{-1}(-\lambda+\lambda_0)fd\lambda\\&=-B_2 f.\end{aligned} \tag{13}$$

Since $\overline{D(A)}=H$, for any $x\in H$, there exists a sequence $\{f_n\}\subset D(A)$ such that $f_n\to x$. Since B_1 and B_2 are bounded, $B_i f_n\to B_i x$, $i=1,2$, $\lambda_0 I-A$ being closed, thus $B_1 x\in D(A)$ and

$$(\lambda_0-L-K)B_1 x=-B_2 x, \text{ for any } x\in H. \tag{14}$$

Morever, for any $f \in D(A)$,

$$(I-(\lambda I-L)^{-1}K)^{-1}(\lambda_0 I-L-K)f$$
$$=(I-(\lambda I-L)^{-1}K)^{-1}(-\lambda+\lambda_0)f+(I-(\lambda I-L)^{-1}K)^{-1}(\lambda I-L-K)f$$

and $(I-(\lambda I-L)^{-1}K)^{-1}(\lambda I-L-K)f$ is analytic in $|\lambda-\lambda_0|\leqslant\delta$, because λ_0 is a pole of $(\lambda I-L-K)^{-1}$ and $(I-(\lambda I-L)^{-1}K)^{-1}$ with the same order. So

$$B_2=-B_1(\lambda_0 I-L-K),\ \text{on } D(A) \tag{15}$$

Therefore for any $x\in R(I-(\lambda_0 I-L)^{-1}K)$, from (14),

$$B_2 x=-(\lambda_0 I-L-K)B_1 x=0 \tag{16}$$

and for any x in $N(I-(\lambda_0 I-L)^{-1}K)$, $x\in D(A)$, from (15)

$$\begin{aligned} B_2 x &= -B_1(\lambda_0 I-L-K)x \\ &= -B_1(\lambda_0 I-L)(I-(\lambda_0 I-L)^{-1}K)x=0 \end{aligned} \tag{17}$$

From(10),(16) and (17), $B_2 x=0$ for all $x\in H$.

By the same process with (14), one can show that

$$B_m=-(\lambda_0 I-A)B_{m-1},\ m=3,4,\cdots \tag{18}$$

Therefore we show λ_0 is of order 1, That is the index of λ_0 is 1.

Theorem 2.2. For any real eigenvalue β_0 of A, its index is 1.
Proof. Because of Theorem 2.1, it is enough to show

$$N(I-K_{\beta_0})\supset N(I-K_{\beta_0})^2$$

For any $f\in N(I-K_{\beta_0})^2$, let $g=(I-K_{\beta_0})f$. If $g\neq 0$, we denote $g^*(\mathbf{r},\Omega)=\overline{g(\mathbf{r},-\Omega)}$. It is obvious by (7) and (8) that

$$\beta_0 g^*-L^*g^*-K^*g^*=0$$

Let $h=K^*g^*$, then $h=K\bar{g}$, and $h=K^*(\beta_0 I-L^*)^{-1}h$, That is to say that $h \in N(I - K^*_{\beta_0})$.

Since $N(I-K^*_{\beta_0}) = R(I-K_{\beta_0})^{\perp}$,

$$(g,h) = (g,K\bar{g}) = 0. \tag{19}$$

Since β_0 is real, and L and K are real operators, we can assume that g is real. So

$$(g,K\bar{g}) = \frac{C}{4\pi}\int_V \left(\int_U g(\mathbf{r},\Omega')d\Omega'\right)^2 d\mathbf{r}.$$

Since $\int_U g(\mathbf{r},\Omega')d\Omega' \neq 0$, otherwise, $Kg=0$ and $g=(\beta_0 I-L)^{-1}Kg=0$, $(g, K\bar{g})>0$. This is a contradiction with (19). So the index of β_0 is one.

Ⅲ. ON COMPLEX EIGENVALUES

In this section, we show that there exists no nonreal eigenvalue in $\{\lambda \in \mathbf{C} \mid \mathrm{Re}\lambda \geq -\Sigma\}$.

Let W be the Hilbert space consisting of the absolutely square integral function on V with a new inner product $\langle \cdot, \cdot \rangle$:

$$\langle \varphi,\psi \rangle = 4\pi\int_V \varphi(\mathbf{r})\,\overline{\psi(\mathbf{r})}d\mathbf{r},\ \varphi,\psi \in W. \tag{20}$$

Then W is a subspace of H.

Lemma 3.1[6]. If $f(\mathbf{r})=f(|\mathbf{r}|)$, $\mathbf{r}\in R^3$ has Fourier transform in W, then

$$\mathbf{f}(|\mathbf{r}-\mathbf{r}'|) = \int_{\mathbf{R}^3} \exp[\mathbf{i}(\mathbf{r}-\mathbf{r}')\mathbf{k}]\mathbf{f}^*(k)d\mathbf{k},\ k = |\mathbf{k}|$$

Where, $f^*(k) = \frac{4\pi}{k}\int_0^\infty rf(r)\sin rk\,dr$

Lemma 3.2. Let I be an integral operator on W defined by:

$$I\varphi = \int_V I(\mathbf{r},\mathbf{r}')\varphi(\mathbf{r}')d\mathbf{r}'$$

Where $I(\mathbf{r},\mathbf{r}')=\frac{\exp[-(\beta+\Sigma-i\alpha)|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|^2}$, α and β are real.

If $\beta+\Sigma>0$, then for any $\varphi\in W$, $\varphi\neq 0$,

(a). $\mathrm{Im}\langle I\varphi,\varphi\rangle>0$, as $\alpha>0$.

(b). $\mathrm{Im}\langle I\varphi,\varphi\rangle<0$, as $\alpha<0$.

Proof. Let

$$f(x)=\frac{\exp[-(\beta+\Sigma-i\alpha)x]}{x^2}$$

If $\beta+\Sigma>0$, then

$$f^*(k)=\frac{4\pi}{k}\int_0^\infty\frac{\exp[-(\beta+\Sigma-i\alpha)x]}{x}\sin kx\,dx$$
$$=\frac{2\pi}{ki}\ln\frac{\beta+\Sigma+(k-\alpha)i}{\beta+\Sigma-(k+\alpha)i}$$

Thus,

$$\mathrm{Im}f^*(k)=-\frac{2\pi}{k}\ln\frac{\sqrt{(\beta+\Sigma)^2+(k-\alpha)^2}}{\sqrt{(\beta+\Sigma)^2+(k+\alpha)^2}}\qquad(21)$$

Therefore, for any $\varphi\in W$, $\varphi\neq 0$, by Lemma 3.1,

$$\mathrm{Im}\langle I\varphi,\varphi\rangle=\mathrm{Im}4\pi\int_V\varphi(\mathbf{r})\int_V f|\mathbf{r}-\mathbf{r}'|\varphi(\mathbf{r}')d\mathbf{r}'d\mathbf{r}$$
$$=4\pi\int_{R^3}\mathrm{Im}f^*(k)d\mathbf{k}\left|\int_V\varphi(\mathbf{r})\exp(-i\mathbf{r}\mathbf{k})d\mathbf{r}\right|^2$$

From (21),

(a). $\mathrm{Im}\langle I\varphi,\varphi\rangle>0$, as $\alpha>0$.

(b). $\mathrm{Im}\langle I\varphi,\varphi\rangle<0$, as $\alpha<0$.

Theorem 3.1. Let $\beta_0=\beta-i\alpha,\alpha\neq 0$ and $\beta>-\Sigma$. Then β_0 is not an eigenvalue of A.

Proof. If $\beta_0=\beta-i\alpha$, $\alpha\neq 0$ and $\beta>-\Sigma$, is an eigenvalue of A, for any $g\in N(I-K_{\beta_0})$, g is a complex function.

Let $g=g_1+ig_2$, $g_1\neq 0$, $g_2\neq 0$, and $\varphi=\varphi_1+i\varphi_2$, where

$$\varphi_1(\mathbf{r}) = \int_U g_1(\mathbf{r},\Omega')d\Omega' \quad \varphi_2(\mathbf{r}) = \int_U g_2(\mathbf{r},\Omega')d\Omega'$$

Note that

$$\varphi = \frac{C}{4\pi}J(\beta_0 I - L)^{-1}\varphi = \frac{C}{4\pi}I\varphi$$

where J is defined in (6)

Since the following equation (22) is impossible by Lemma 3.2,

$$0 = \mathrm{Im}\langle\varphi,\varphi\rangle = (C/4\pi)\mathrm{Im}\langle I\varphi,\varphi\rangle \tag{22}$$

therefore we complete the proof of Theorem 3.1.

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