

Random Walk

Symmetric Random Walk

Let

$$X_i = \begin{cases} +1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2, \end{cases} \quad i = 1, 2, \dots$$

For $n \geq 1$, let $W_n = X_1 + X_2 + \dots + X_n$, with $W_0 = 0$. Then $\{W_n\}_n$ is called a symmetric random walk. A continuous trajectory is shown in Figure 1, where discrete states (points) are connected by linear interpolation:

$$W_t = \begin{cases} X_1 + X_2 + \dots + X_t, & \text{if } t \text{ is an integer,} \\ W_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)X_{\lfloor t \rfloor + 1}, & \text{otherwise.} \end{cases}$$

where $\lfloor t \rfloor$ is the floor of t .

```
set.seed(12345)
n <- 100
t <- seq(0, n, 1)
xi <- sample(c(-1, 1), replace=TRUE, size=length(t)-1, prob = c(0.5, 0.5))
Wn <- c(0, cumsum(xi))
df <- data.frame(x=t, y=Wn)
df$xend <- c(df$x[2:nrow(df)], NA)
df$yend <- c(df$y[2:nrow(df)], NA)
p <- ggplot(df, aes(x=x, y=y)) + geom_point() + geom_segment(aes(x=x, xend=xend,
  y=y, yend=yend)) +
  theme_bw() # white background with frame
p
## Warning: Removed 1 rows containing missing values (geom_segment).
```

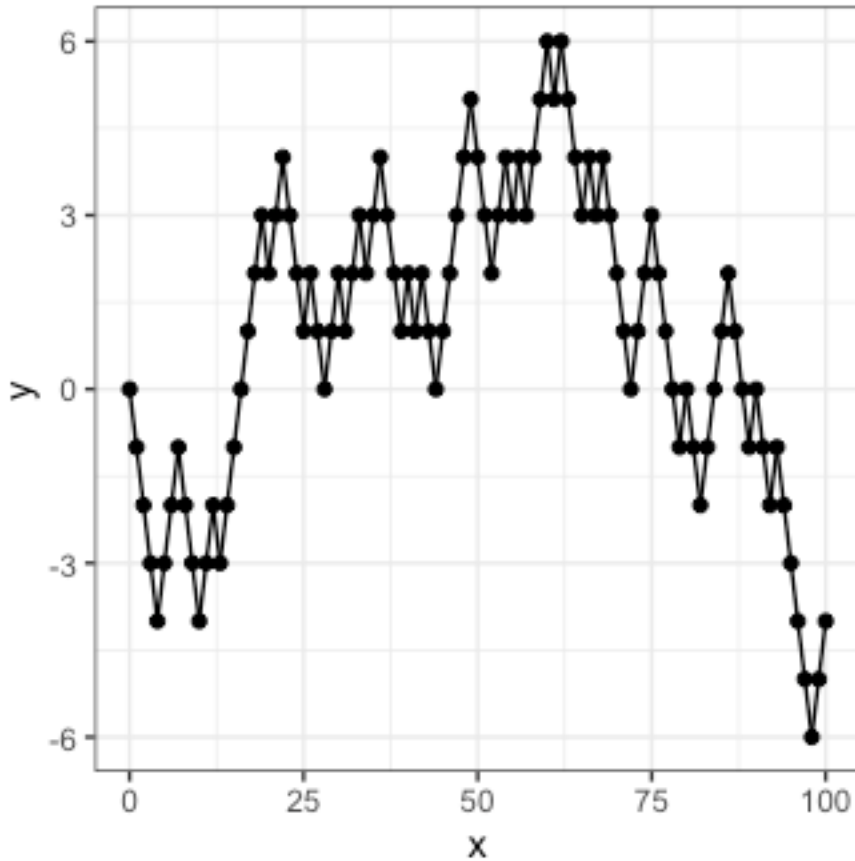


Figure 1: A trajectory of a simple symmetric random walk.

Note 2. The simple symmetric random walk has independent increments, i.e. for integers $0 < q < r < s < t$, $W_t - W_s$ and $W_r - W_q$ are independent.

Note 3. For all integers $0 < s < t$, the distribution of $W_t - W_s$ and W_{t-s} is the same since both are a function of $t - s$ iid X_i .

Scaled Random Walk

Let $\{W_n\}_{n \geq 1}$ be a symmetric random walk with $W_0 = 0$. Then $S^{(n)}(t) = \frac{1}{\sqrt{n}} W_{[nt]}$ is called scaled random walk, or the n -th level approximation of a Brownian motion. A trajectory is shown in Figure 2.

```

set.seed(12345)
k <- 8
n <- 2^k
t <- seq(0, 10, length=(n+1))
xi <- sample(c(-1, 1), replace=TRUE, size=n, prob = c(0.5, 0.5))
Sn <- c(0, cumsum(xi)/sqrt(n))
df <- data.frame(x=t, y=Sn)
df$xend <- c(df$x[2:nrow(df)], NA)
df$yend <- c(df$y[2:nrow(df)], NA)
p <- ggplot(df, aes(x=x, y=y)) + geom_point() + geom_segment(aes(x=x, xend=xend,
y=y, yend=yend)) +
  theme_bw() # white background with frame
p
## Warning: Removed 1 rows containing missing values (geom_segment).

```

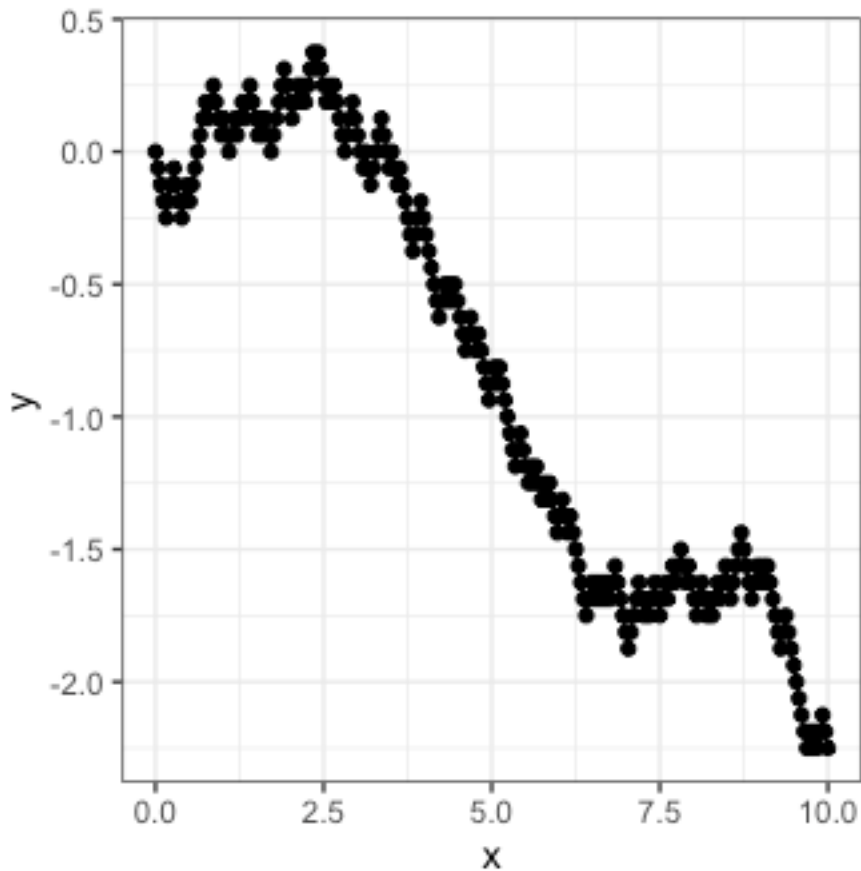


Figure 2: A trajectory of a scaled random walk.