XX Oscillation Models

XX.1 Complex number

For a complex variable $z \in \mathbb{C}$, the power series expansion of e^z is

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \frac{x^n}{n!} + \cdots \tag{XX.1}$$

For a real variable $x \in \mathbb{R}$, the power series expansions of $\sin x$ and $\cos x$ are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \tag{XX.2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \tag{XX.3}$$

From Eq. XX.1, Eq. XX.2 and Eq. XX.1, we can obtain the so-called **Euler's formula**:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}; \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{XX.4}$$

The Euler's formula establishes the fundamental relationship between trigonometric functions and exponential functions.

Next, we introduce two very useful propositions.

Proposition XX.1. $C \in \mathbb{C}$ is a real number if and only if $C = \bar{C}$.

Proposition XX.2. Let $C_1,\ C_2 \in \mathbb{C}$ be complex numbers, $a,\ b,\ t \in \mathbb{R}$ real numbers. If

$$y = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} \tag{XX.5}$$

is real, then y can be expressed as

$$y = Ae^{at}\cos(bt + \phi) \tag{XX.6}$$

where A is a positive real number and ϕ is a real number.

Proof. If $y(t)$ is real, then $y = \overline{y}$, i.e.

$$C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} = \bar{C_1} e^{(a-bi)t} + \bar{C_2} e^{(a+bi)t}$$

or

$$(C_1 - \bar{C_2}) e^{(a+bi)t} + (C_2 - \bar{C_1}) e^{(a-bi)t} = 0$$

This yields $C_1 = \bar{C_2}$.

Let $C_1 = C_R + iC_I$, $C_R, C_I \in \mathbb{R}$. y can be rewritten as

$$\begin{aligned} y &= C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} \\ &= (C_R + iC_I) e^{(a+bi)t} + (C_R - iC_I) e^{(a-bi)t} \\ &= e^{at}\{C_R(e^{ibt} + e^{-ibt}) + iC_I(e^{ibt} - e^{-ibt})\} \\ &= 2e^{at}(C_R \cos(bt) - C_I \sin(bt)) \\ &= 2\sqrt{C_R^2 + C_I^2}\; e^{at}\left(\frac{C_R}{\sqrt{C_R^2 + C_I^2}}\cos(bt) - \frac{C_I}{\sqrt{C_R^2 + C_I^2}}\sin(bt)\right) \\ &= Ae^{at}\cos(bt + \phi) \end{aligned}$$

XX.2 Free oscillation

As shown in Fig. XX.1, a small ball with mass m is attached on one end of a massless spring with a force constant k while the other end of the spring is fixed. The equilibrium position of the ball is at $x = 0$. The ball is gently and slowly pulled to the position $x = x_0$, and then released. Friction is negligible. According to Newton's law and Hooke's law, the motion of the ball can be described as

$$m\frac{d^2x(t)}{dt^2} = -kx(t) \tag{XX.7}$$

The general solution to Eq. XX.7 is

$$x(t) = A\cos(\omega t + \varphi) \tag{XX.8}$$

where $\omega = \sqrt{\frac{k}{m}}$ is called the angular frequency with the unit $radians/second$; $T = \frac{2\pi}{\omega}$ is the period with the unit $second$; $f = 1/T$ is the frequency with the unit Hz; φ is the phase angle with the unit radians; A is the amplitude with the unit $meter$.

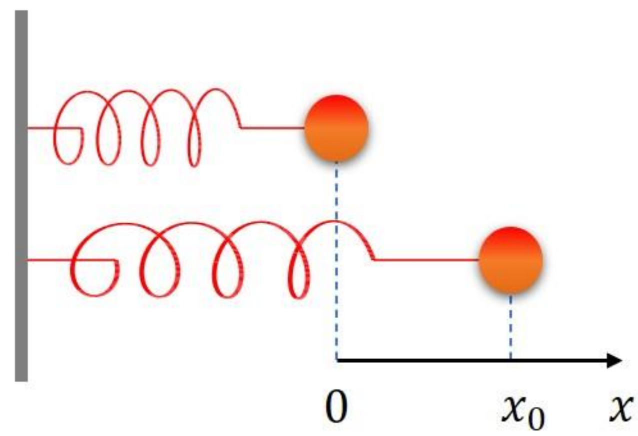

Figure XX.1. Schematic drawing of a free oscillator. A mass is attached on one end of a spring whereas the other end of the spring is fixed.

In order to make a connection between eigenvalues and frequencies, we rewrite Eq. XX.1 in a matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} \tag{XX.9}$$

where $y(t) = \dot{x}(t)$, and

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

The matrix $\mathcal{A}$ has eigenvalues $\pm i\omega$ and corresponding engenvectors $[1, \pm i\omega]^T$, $i^2 = -1$. Therefore, the general solution to Eq. XX.9 is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{i\omega t}\begin{bmatrix} 1 \\ i\omega \end{bmatrix} + c_2 e^{-i\omega t}\begin{bmatrix} 1 \\ -i\omega \end{bmatrix} \tag{XX.10}$$

Thus, using **Proposition XX.2** yields the real solution

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} = A\cos(\omega t + \varphi) \tag{XX.11}$$

Using the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, we obtain the solution to Eq. XX.7:

$$x(t) = x_0\cos(\omega t) \tag{XX.12}$$

Here we notice that the positive imaginary part of the eigenvalues of the matrix $\mathcal{A}$ is the angular frequency.

XX.3 Damped oscillation

Taking friction into account, we need to include a drag force which is anti-proportional to the velocity of the ball in Eq. XX.7. In this case, the energy of the oscillator leaks to its environment in the form of heat. This effect is called energy dissipation. Because of energy dissipation, the free oscillation will decay with time, which is called damped oscillation. A damped oscillation system can be described as

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \tag{XX.13}$$

where the constant b is called the drag coefficient.

Let $\omega_0 = \sqrt{\frac{k}{m}}$ and $\gamma = \frac{b}{2m}$. Then Eq. XX.13 becomes

$$\frac{d^2x}{dt^2} + \omega_0^2 x + 2\gamma\frac{dx}{dt} = 0 \tag{XX.14}$$

Substituting $z = Ae^{i(pt+\alpha)}$ into the equation Eq. XX.14, we obtain

$$-p^2 + \omega_0^2 + i2p\gamma = 0 \tag{XX.15}$$

If p is a real number, $p = 0$ and $\omega_0 = 0$, which is not true by the definition of ω_0. So p must be an imaginary number. Suppose $p = \omega' + is$.

First, we consider the case of $\gamma < \omega_0$, which is called under-damping. From Eq. XX.15, we obtain

$$\begin{cases} s = \gamma \\ \omega' = \sqrt{\omega_0^2 - \gamma^2} \end{cases}$$

and

$$z = Ae^{i(pt+\alpha)} = Ae^{i(\omega' t + ist + \alpha)} = Ae^{-st}e^{i(\omega' t + \alpha)} \tag{XX.16}$$

Thus Eq. XX.13 has a real solution

$$x(t) = Ae^{-\gamma t}\cos(\omega' t + \alpha) \tag{XX.17}$$

where A and α are determined by initial and boundary conditions.

In order to make the connection between eigenvalues and frequencies, we rewrite Eq. XX.14 in a matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{A}_d \begin{bmatrix} x \\ y \end{bmatrix} \tag{XX.18}$$

where $y(t) = \dot{x}(t)$, and

$$\mathcal{A}_d = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{bmatrix}$$

The matrix $\mathcal{A}_d$ has eigenvalues $-\gamma \pm i\omega'$ and corresponding eigenvectors $[1, -\gamma \pm i\omega']^T$. Therefore, the general solution to Eq. XX.18 is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{-\gamma/2}\{c_1 e^{i\omega' t}\begin{bmatrix} 1 \\ i\omega' \end{bmatrix} + c_2 e^{-i\omega' t}\begin{bmatrix} 1 \\ -i\omega' \end{bmatrix}\} \tag{XX.19}$$

Thus, from **Proposition XX.2,** the real solution is

$$x(t) = Ae^{-\gamma t}\cos(\omega' t + \alpha) \tag{XX.20}$$

Using the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, we obtain the solution

$$x(t) = x_0 e^{-\gamma t}\cos(\omega' t) \tag{XX.21}$$

Here we notice that the positive imaginary part of the eigenvalues of the matrix $\mathcal{A}_d$ is the angular frequency.

Overdamping

When $\gamma > \omega_0$, it is called overdamping. In this case,

$$\omega' = \pm(\gamma^2 - \omega_0^2)^{1/2} \tag{XX.22}$$

The general solution for overdamping is

$$x(t) = e^{-\gamma t}\left(c_1 e^{(\gamma^2-\omega_0^2)^{1/2} t} + c_2 e^{-(\gamma^2-\omega_0^2)^{1/2} t}\right) \tag{XX.23}$$

Critical damping

When $\omega_0 = \gamma$, it is called critical damping. In this case, the general solution is

$$x = (A + Bt)e^{-\gamma t} \tag{XX.24}$$

XX.4 Forced oscillation

If the oscillation is driven by an external force $F_0\cos(\omega_D t)$, Eq. XX.13 needs to include the external force effect:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0\cos(\omega_D t) \tag{XX.25}$$

Consider Eq. XX.25 as the real part of z satisfying

$$\frac{d^2z}{dt^2} + \omega_0^2 z + 2\gamma\frac{dz}{dt} = \frac{F_0}{m}e^{j\omega_D t} \tag{XX.26}$$

We have derived the general solution Eq. XX.20 to the homogeneous equation of Eq. XX.26. Now we need find a particular solution to the inhomogeneous equation Eq. XX.26. Because ω_D will be the dominant frequency after a long time, we assume a steady state solution has the form of $z = Ae^{j(\omega_D t-\delta)}$. Substituting $z = Ae^{j(\omega_D t-\delta)}$ into Eq. XX.26, we obtain

$$-\omega_D^2 z + \omega_0^2 z + 2\gamma\omega_D jz = \frac{F_0}{m}e^{j\omega_D t} \tag{XX.27}$$

Cancelling $e^{j(\omega_D t-\delta)}$ in both side yields

$$A(-\omega_D^2 + \omega_0^2 + 2\gamma\omega_D j) = \frac{F_0}{m}\cos(\delta) + j\frac{F_0}{m}\sin(\delta)$$

or

$$\begin{cases} A(-\omega_D^2 + \omega_0^2) = \dfrac{F_0}{m}\cos(\delta) \\ 2A\gamma\omega_D = \dfrac{F_0}{m}\sin(\delta) \end{cases}$$

Using $sin^2(\delta) + cos^2(\delta) = 1$, we obtain

$$A = \frac{F_0}{m}\left[(\omega_0^2 - \omega_D^2)^2 + \frac{b^2\omega_D^2}{m^2}\right]^{-1/2}$$

$$\tan\delta = \frac{2\gamma\omega_D}{\omega_0^2 - \omega_D^2}$$

Equation Eq. XX.26 has a real solution

$$x = A\cos(\omega_D t - \delta) \qquad \text{(XX.28)}$$

The solution does not depend on any initial conditions. At $\omega_{max} = (\omega_0^2 - 2\gamma^2)^{1/2}$, $A_{max} = \frac{F_0}{2k}\frac{Q}{(1-\frac{1}{Q^2})^{1/2}}$, where $Q = \frac{\omega_0}{\gamma}$ is called the quality factor.

The general solution of the driven system Eq. XX.25 is the sum of Eq. XX.28 and Eq. XX.20:

$$x = A\cos(\omega_D t - \delta) + Xe^{-\gamma t}\cos(\omega' t + \alpha) \qquad \text{(XX.29)}$$

The first term is the steady state solution, the second term is the transient solution which will eventually decay.

Consider the work done to the system by the external force:

$$dW = \vec{F} \cdot \overrightarrow{dx} \qquad \text{(XX.30)}$$

and the power

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{\overrightarrow{dx}}{dt} \qquad \text{(XX.31)}$$

Applying to the steady state Eq. XX.28 of the driven system since the transition state will die out, we obtain

$$P = Fv = F_0\cos(\omega_D t)[-\omega_D A\sin(\omega_D t - \delta)] \qquad \text{(XX.32)}$$

The average P in a cycle is

$$
\begin{aligned}
\bar{P} &= \frac{1}{T}\int_0^T P\,(t)dt \\
&= -\omega_D A F_0 \frac{1}{T}\int_0^T \cos\,(\omega_D t)[\sin(\omega_D t)\cos\delta - \cos(\omega_D t)\sin\delta)]dt \\
&= \frac{F_0}{2}\omega_D A\,\sin\delta \\
&= \frac{F_0}{2} A \frac{2{\omega_D}^2\gamma}{\sqrt{(\omega_0^2 - {\omega_D}^2)^2 + (2\omega_D\gamma)^2}} \\
&= \frac{F_0^2\gamma}{m}\left[\left(\frac{\omega_0^2}{\omega_D} - \omega_D\right)^2 + 4\gamma^2\right]^{-1}
\end{aligned}
\qquad \text{(XX.33)}
$$

It is easy to see that

1) If $\gamma \to \infty$, $\bar{P} \to 0$. It means that if the friction is very large, the external force does no useful work.
2) If $m \to \infty$, $\bar{P} \to 0$. It means that if the mass is very large, the external force does no useful work.
3) If $F_0 \to 0$, $\bar{P} \to 0$. It means that no force.
4) If $\omega_D \to \infty$, $\bar{P} \to 0$. It means that the driving force oscillates too fast to do useful work.
5) If $\omega \to 0$, $\bar{P} \to 0$. It means that the driving force oscillates too slow to do useful work.
6) If $\omega \to \omega_0$, $\bar{P} \to \bar{P}_{max} = \frac{F_0^2}{4m\gamma}$. It means the driving force does maximum useful work to the system. This is called the resonance phenomena.

XX.5 Electron oscillation

Consider an atom in an oscillating electric field $E = E(t)$ along the z-direction. We assume a particular electron of this atom is oscillating around $z = 0$. The motion of the electron can be described as

$$
\ddot{z} + 2\eta\dot{z} + \Omega_0^2 z = a(t) \quad \text{where} \quad a(t) = -\frac{e}{m}E(t) \qquad \text{(XX.34)}
$$

where m and $-e$ are the electron's mass and charge, $m\Omega_0^2$ is the spring constant of the linear oscillator, $2m\eta$ the friction coefficient.

XX.5.1 Green's function

We assume there is a linear relationship between position z and acceleration a:

$$z(t) = \int ds\Gamma(t, s)a(s) \qquad \text{(XX.35)}$$

where Γ is a Green's function.

Because the coefficients of Eq. XX.34 do not depend on time t, we have $\Gamma(t, s) = \Gamma(t - s)$, and thus

$$z(t) = \int ds\Gamma(t - s)a(s) = \int d\tau\Gamma(\tau)a(t - \tau) \qquad \text{(XX.36)}$$

The displacement z is caused by the acceleration a. Causes must be earlier that their effect. Therefore

$$z(t) = \int_{-\infty}^{t} \Gamma(t - s)a(s)ds = \int_{0}^{\infty} \Gamma(\tau)a(t - \tau)d\tau \qquad \text{(XX.37)}$$

Taking derivative with respect to time t in Eq. XX.37, we obtain:

$$\dot{z}(t) = \Gamma(0)a(t) + \int_{-\infty}^{t} d\,s\dot{\Gamma}(t - s)a(s) \qquad \text{(XX.38)}$$

We differentiate Eq. XX.38 with respect to time t:

$$\ddot{z}(t) = \Gamma(0)\dot{a}(t) + \dot{\Gamma}(0)a(t) + \int_{-\infty}^{t} d\,s\ddot{\Gamma}(t - s)a(s) \qquad \text{(XX.39)}$$

Inserting Eq. XX.38 and Eq. XX.39 into Eq. XX.34 yields

$$\ddot{\Gamma} + 2\eta\dot{\Gamma} + \Omega_0^2\Gamma = 0 \qquad \text{(XX.40)}$$

and

$$\Gamma(0) = 0 \quad \text{and} \quad \dot{\Gamma}(0) = 1 \qquad \text{(XX.41)}$$

From Section XX.3, we know if $\eta/\Omega_0 < 1$, Eq. XX.40 has a general solution:

$$\Gamma(\tau) = c_1 e^{(-\eta+i\Omega)t} + c_2 e^{(-\eta-i\Omega)t}, \qquad \Omega = \sqrt{\Omega_0^2 - \eta^2} \tag{XX.42}$$

Using initial conditions Eq. XX.41, we obtain

$$\Gamma(\tau) = \frac{1}{\Omega} e^{-\eta\tau} \sin\Omega\tau \tag{XX.43}$$

Substituting Eq. XX.43 into Eq. XX.37 results in the solution

$$z(t) = \frac{-e}{m}\int_0^\infty \Gamma(\tau)E(t-\tau)d\tau = \frac{-e}{m\Omega}\int_0^\infty d\,\tau E(t-\tau)e^{-\eta\tau}\sin\Omega\tau \tag{XX.44}$$

XX.5.2 Susceptibility

The dipole moment of the above considered electron is $p = -ez$. If there are N of them per unit volume, the polarization is $P(t) = Np$。 Using Eq. XX.44, we obtain:

$$\begin{aligned} P(t) \quad &= Np = -eNz(t) = \frac{Ne^2}{m}\int_0^\infty d\,\tau\Gamma(\tau)E(t-\tau) \\ &= \int_{-\infty}^\infty d\,\tau[\frac{Ne^2}{m}H(\tau)\Gamma(\tau)]E(t-\tau) \end{aligned} \tag{XX.45}$$

where $H(\tau)$ is the Heaviside step function.

Fourier transforming Eq. XX45 results in

$$\hat{P}(\omega) = \epsilon_0\chi(\omega)\hat{E}(\omega) \tag{XX.46}$$

where

$$\hat{E}(\omega) = \int d\omega\; e^{i\omega t}E(t) \tag{XX.47}$$

and

$$
\begin{aligned}
\chi(\omega) &= \frac{Ne^2}{m\epsilon_0}\int_0^\infty d\,\tau e^{i\omega\tau}\ \Gamma(\tau) \\
&= \frac{Ne^2}{m\epsilon_0}\int_0^\infty d\,\tau e^{i\omega\tau}\frac{1}{2i\Omega}e^{-\eta\tau}(e^{i\Omega\tau}-e^{-i\Omega\tau}) \\
&= \frac{Ne^2}{m\epsilon_0}\frac{1}{(\Omega+\omega+i\eta)(\Omega-\omega-i\eta)} \\
&= \frac{Ne^2}{m\epsilon_0}\frac{1}{\Omega_0^2-\omega^2-2i\eta\omega} \\
&= \frac{Ne^2}{m\epsilon_0}\frac{\Omega_0^2-\omega^2+2i\eta\omega}{(\Omega_0^2-\omega^2)^2+4\eta^2\omega^2}
\end{aligned} \qquad \text{(XX.48)}
$$

$\chi = \chi(\omega)$ is called the susceptibility of the material under study. It is a function of angular frequency. Its imaginary part

$$
Im\,\chi(\omega) = \frac{Ne^2}{m\epsilon_0}\frac{2\eta\omega}{(\Omega_0^2-\omega^2)^2+4\eta^2\omega^2} \qquad \text{(XX.49)}
$$

has the following properties: Eq. 15 或 Eq. 16 有下面的基本特征：

- $Im\,\chi(\omega) \to 0$, as $\omega \to \infty$.
- $Im\,\chi(\omega) > 0$
- Resonance frequency $\omega \approx \Omega_0$.